MATH 303 – Measures and Integration
Quiz Questions

Note: Some of the questions may be in a slightly different order than they appeared when you took the quiz. This is because some portions of the quiz were set to display the questions in a random order.

1 Domains of Measures, Premeasures, and Outer Measures

Problem 1. Match the type of object with the domain on which it is defined.

- Premeasure: ________________
- Measure: ________________
- Outer Measure: ________________

Options: power set, algebra, σ-algebra

2 Properties of Measures, Premeasures, and Outer Measures

Problem 2. Select **all** of the properties that must be satisfied by a **measure**.

☐ assigns a value of 0 to the empty set

☐ nonnegative

☐ monotone

☐ countably additive

☐ countably subadditive

Problem 3. Select **all** of the properties that must be satisfied by a **premeasure**.

☐ assigns a value of 0 to the empty set

☐ nonnegative

☐ monotone

☐ countably additive

☐ countably subadditive

Problem 4. Select **all** of the properties that must be satisfied by an **outer measure**.

☐ assigns a value of 0 to the empty set

☐ nonnegative

☐ monotone

☐ countably additive

☐ countably subadditive

3 Measurable Functions

Problem 5. Let $(X, \mathcal{B})$, $(Y, \mathcal{C})$, and $(Z, \mathcal{D})$ be measurable spaces, and let $f : X \to Y$ and $g : Y \to Z$. **True or False**: If f and g are measurable functions, then the composition $g \circ f$ is also a measurable function.

☐ True

☐ False

Problem 6. Let $(X, \mathcal{B})$, $(Y, \mathcal{C})$, and $(Z, \mathcal{D})$ be measurable spaces, and let $f : X \to Y$ and $g : Y \to Z$. **True or False**: If $g \circ f$ is a measurable function, then f and g are measurable functions.

☐ True

☐ False

4 Lebesgue Measurable Functions

Problem 7. We say that a real-valued function $f : \mathbb{R} \to \mathbb{R}$ is **Lebesgue-measurable** if it is measurable as a map from the measurable space $(\mathbb{R}, \mathcal{M})$ to the measurable space $(\mathbb{R}, \mathrm{Borel}(\mathbb{R}))$, where $\mathcal{M}$ is the σ-algebra of Lebesgue-measurable sets. This means (**fill in the blanks** by choosing the appropriate option at the bottom):

For every $\boxed{}$ set $E \subseteq \mathbb{R}$, the set $\boxed{}$ is $\boxed{}$.

Options: Borel, Lebesgue-measurable, $f(E)$, $f^{-1}(E)$

Problem 8. We say that a real-valued function $f : \mathbb{R} \to \mathbb{R}$ is **Lebesgue-measurable** if it is measurable as a map from the measurable space $(\mathbb{R}, \mathcal{M})$ to the measurable space $(\mathbb{R}, \mathrm{Borel}(\mathbb{R}))$, where $\mathcal{M}$ is the σ-algebra of Lebesgue-measurable sets.

Let $f, g : \mathbb{R} \to \mathbb{R}$. **True or False**: If f and g are Lebesgue-measurable functions, then the composition $g \circ f : \mathbb{R} \to \mathbb{R}$ is also Lebesgue-measurable.

☐ True

☐ False

5 Integration

Problem 9. The monotone convergence theorem is a theorem about a sequence of functions $(f_n)_{n\in\mathbb{N}}$ satisfying certain properties. Select **all** of the properties of $(f_n)_{n\in\mathbb{N}}$ that are part of the hypothesis of the monotone convergence theorem.

- ☐ f_n is continuous for each n
- ☐ f_n is differentiable for each n
- ☐ f_n is a simple function for each n
- ☐ f_n is a measurable function for each n
- ☐ f_n is nonnegative for each n
- ☐ f_n is an increasing function for each n
- ☐ $(f_n)_{n\in\mathbb{N}}$ is an increasing sequence of functions
- ☐ $(f_n)_{n\in\mathbb{N}}$ is a decreasing sequence of functions

Problem 10. Let $(f_n)_{n\in\mathbb{N}}$ be an increasing sequence of extended real-valued measurable functions. **True or False**: The pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$ exists for every x, and f is a measurable function.

- ☐ True
- ☐ False

Problem 11. Fill in the blanks to complete the statement of the monotone convergence theorem.

Monotone Convergence Theorem: Let $(X, \mathcal{B}, \mu)$ be a measure space. Let $(f_n)_{n\in\mathbb{N}}$ be a(n) $\boxed{\text{decreasing/increasing}}$ sequence of $\boxed{\text{nonnegative/real-valued}}$ $\boxed{\text{differentiable/simple/continuous/measurable}}$ functions on X. Then

$$\int_X \lim_{n\to\infty} f_n \, d\mu \ \boxed{\leq / = / \geq}\ \lim_{n\to\infty} \int_X f_n \, d\mu.$$

Problem 12. Fatou's lemma is a statement about a sequence of functions $(f_n)_{n\in\mathbb{N}}$ satisfying certain properties. Select **all** of the properties of $(f_n)_{n\in\mathbb{N}}$ that are part of the hypothesis of Fatou's lemma.

- ☐ f_n is continuous for each n
- ☐ f_n is a simple function for each n
- ☐ f_n is a measurable function for each n
- ☐ f_n is nonnegative for each n
- ☐ $(f_n)_{n\in\mathbb{N}}$ is an increasing sequence of functions

☐ $(f_n)_{n\in\mathbb{N}}$ is a decreasing sequence of functions

Problem 13. Fill in the blank to complete the statement of Fatou's Lemma.

Fatou's Lemma: Let $(X, \mathcal{B}, \mu)$ be a measure space. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative measurable functions on X. Then

$$\int_X \liminf_{n\to\infty} f_n \, d\mu \ \boxed{\leq / = / \geq} \ \liminf_{n\to\infty} \int_X f_n \, d\mu.$$

Problem 14. Let $(X, \mathcal{B}, \mu)$ be a measure space, and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of complex-valued integrable functions on X. **True or False**: If the pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$ exists (as a complex number) for every $x \in X$, then f is integrable and $\int_X f \, d\mu = \lim_{n\to\infty} \int_X f_n \, d\mu$.

☐ True

☐ False

Problem 15. Fill in the blanks to complete the statement of the dominated convergence theorem.

Dominated Convergence Theorem: Let $(X, \mathcal{B}, \mu)$ be a measure space. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of integrable functions on X. Suppose that $(f_n)_{n\in\mathbb{N}}$ converges almost everywhere to a measurable function $f : X \to \mathbb{C}$, and there exists a nonnegative $\boxed{\text{measurable/integrable/simple/bounded}}$ function $g : X \to [0, \infty)$ such that $\boxed{\left|\int_X f_n \, d\mu\right| \leq \int_X g \, d\mu \ / \ \int_X |f_n| \, d\mu \leq \int_X g \, d\mu \ / \ |f_n| \leq g \text{ a.e.}}$ for each n. Then

$$\int_X f \, d\mu \ \boxed{\leq / = / \geq} \ \lim_{n\to\infty} \int_X f_n \, d\mu.$$

6 π-Systems and λ-Systems

Problem 16. Select all properties that must be satisfied by a **π-system** $\mathcal{P}$ on a set X.

☐ contains the empty set

☐ contains X

☐ closed under finite intersections

☐ closed under finite unions

☐ closed under complements

☐ closed under relative complements: if $E, F \in \mathcal{P}$ and $E \subseteq F$, then $F \setminus E \in \mathcal{P}$

☐ closed under countable unions

☐ closed under countable disjoint unions

☐ closed under countable increasing unions

Problem 17. Select all properties that must be satisfied by a **λ-system** $\mathcal{L}$ on a set X.

☐ contains the empty set

☐ contains X

☐ closed under finite intersections

☐ closed under finite unions

☐ closed under complements

☐ closed under relative complements: if $E, F \in \mathcal{L}$ and $E \subseteq F$, then $F \setminus E \in \mathcal{L}$

☐ closed under countable unions

☐ closed under countable disjoint unions

☐ closed under countable increasing unions

Problem 18. For each example below, select the correct option that applies **in general**. Click and drag the correct answer from the options at the bottom to fill in each blank.

- The family of open subsets of a topological space is ________________.
- The family of open neighborhoods of a point in a topological space is ________________.
- The family of left-open, right-closed intervals is ________________.
- A σ-algebra is ________________.
- The family of sets $\{E \in \mathcal{B} : \mu(E) = \nu(E)\}$ on which two probability measures μ and ν on a measurable space $(X, \mathcal{B})$ agree is ________________.
- The family $\{E \in \text{Borel}(\mathbb{R}) : \mu(E) = \nu(E)\}$ of Borel sets on which two Lebesgue–Stieltjes measures μ and ν agree is ________________.
- The family $\{E \in \mathcal{B} : \mu(E) = 0\}$ of null sets in a measure space $(X, \mathcal{B}, \mu)$ is ________________.
- The family $\{E \in \mathcal{B} : \mu(X \setminus E) = 0\}$ of co-null sets in a measure space $(X, \mathcal{B}, \mu)$ is ________________.
- The family $\{E \in \mathcal{B} : \mu(E) = 0 \text{ or } \mu(X \setminus E) = 0\}$ of null and co-null sets in a measure space $(X, \mathcal{B}, \mu)$ is ________________.
- The family $\{E \subseteq X : E \text{ or } X \setminus E \text{ is finite}\}$ of finite and co-finite subsets of an infinite set X is ________________.
- The family $\{E \subseteq X : E \text{ or } X \setminus E \text{ is countable}\}$ of countable and co-countable subsets of an uncountable set X is ________________.

Options: a π-system, a λ-system, both a π-system and a λ-system, neither a π-system nor a λ-system

Problem 19. Complete the statement of the π-λ theorem.

π-λ Theorem: Let X be a set, $\mathcal{P}$ a π-system on X, and $\mathcal{L}$ a λ-system on X. Suppose $\mathcal{P}\boxed{\subseteq / = / \supseteq}\mathcal{L}$. Then $\boxed{\mathcal{P} \text{ is a } \sigma\text{-algebra}/\mathcal{L} \text{ is a } \sigma\text{-algebra}/\sigma(\mathcal{P}) \subseteq \mathcal{L}/\sigma(\mathcal{P}) = \mathcal{L}/\sigma(\mathcal{P}) \supseteq \mathcal{L}}$.

The π-λ theorem is useful for proving $\boxed{\text{existence/uniqueness}}$ of measures satisfying certain properties.

7 Radon Measures

Problem 20. Out of the following list of topological spaces, select **all** options that are **locally compact Hausdorff spaces**.

☐ Discrete spaces

☐ The rational numbers with the subspace topology inherited from the standard topology on $\mathbb{R}$

☐ Euclidean space $\mathbb{R}^d$ for $d \in \mathbb{N}$

☐ Compact metric spaces

☐ The space $C[0,1]$ of complex-valued continuous functions $f : [0,1] \to \mathbb{C}$ with the topology of uniform convergence

Problem 21. Let X be an LCH space. A **Radon measure** is a Borel measure $\mu : \text{Borel}(X) \to [0,\infty]$ satisfying 3 properties (click and drag the correct answer from the options at the bottom to fill in each blank):

1. locally finite: the measure of every $\boxed{}$ set is finite
2. outer regular on $\boxed{}$ sets: if E is a(n) $\boxed{}$ set, then
$$\mu(E) = \boxed{}\{\mu(F) : F \text{ is a(n) } \boxed{} \text{ set and } E\boxed{}F\}.$$
3. inner regular on $\boxed{}$ sets: if E is a(n) $\boxed{}$ set, then
$$\mu(E) = \boxed{}\{\mu(F) : F \text{ is a(n) } \boxed{} \text{ set and } E\boxed{}F\}.$$

Options: open, closed, compact, Borel; $\subseteq$, $=$, $\supseteq$; sup, inf

Problem 22. True or False: Every locally finite Borel measure on $\mathbb{R}$ is a Radon measure.

☐ True

☐ False